# Automorphisms and forms of simple infinite-dimensional linearly compact Lie superalgebras

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*To Dmitri V. Alekseevski on his 65th birthday*

**Abstract**

We describe the group of continous automorphisms of all simple infinite-dimensional linearly compact Lie superalgebras and use it in order to classify $\mathbb{F}$-forms of these superalgebras over any field $\mathbb{F}$ of characteristic zero.

*Keywords:* Linearly compact Lie superalgebra, $\mathbb{F}$-form, Galois cohomology.

# Introduction

In our paper [2] we classified all maximal open subalgebras of all simple infinite-dimensional linearly compact Lie superalgebras $S$ over an algebraically closed field $\bar{\mathbb{F}}$ of characteristic zero, up to conjugation by the group $G$ of inner automorphisms of the Lie superalgebra $DerS$ of continuous derivations of $S$. An

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immediate corollary of this result is Theorem 11.1 of [2], which describes, up to conjugation by $G$, all maximal open subalgebras of $S$, which are invariant with respect to all inner automorphisms of $S$. Using this result and an explicit description of $DerS$ (see [10, Proposition 6.1] and its corrected version [2, Proposition 1.8]), we derive the classification of all maximal among the open subalgebras of $S$, which are $AutS$-invariant, where $AutS$ is the group of all continuous automorphisms of $S$ (Theorem 3.4). Such a subalgebra $S_0$ always exists, and in most of the cases it is unique (also, in most of the cases it is a subalgebra of minimal codimension). Picking a subspace $S_{-1}$ of $S$, which is minimal among $AutS$-invariant subspaces, properly containing $S_0$, we can construct the Weisfeiler filtration (see e.g. [2] or [10]). Then it is easy to see that

$$AutS = \mathcal{U} \rtimes AutgrS, \tag{1}$$

where $\mathcal{U}$ is a normal prounipotent subgroup consisting of automorphisms of $S$ inducing an identity automorphism of $GrS$, and $AutgrS$ is a subgroup of a (finite-dimensional) algebraic group of automorphisms of $GrS$, preserving the grading.

We list all the groups $AutgrS$, along with their (faithful) action on $Gr_{-1}S$, in Table 1. This leads to the following description of the group $AutS$:

$$AutS = InautS \rtimes A, \tag{2}$$

where $InautS$ is the subgroup of all inner automorphisms of $S$ and $A$ is a closed subgroup of $AutgrS$, listed in Corollary 4.3.

Let $\mathbb{F}$ be a subfield of $\bar{\mathbb{F}}$, whose algebraic closure is $\bar{\mathbb{F}}$, and fix an $\mathbb{F}$-form $S^{\mathbb{F}}$ of $S$, i.e., a Lie superalgebra over $\mathbb{F}$, such that $S^{\mathbb{F}} \otimes_{\mathbb{F}} \bar{F} \cong S$. Then all $\mathbb{F}$-forms of $S$, up to isomorphism, are in a bijective correspondence with $H^1(Gal, AutS)$, where $Gal = Gal(\bar{\mathbb{F}}/\mathbb{F})$ (see e.g. [13]). Since the first Galois cohomology of a prounipotent algebraic group is trivial (see e.g. [13]), we conclude, using the cohomology long exact sequence, that

$$H^1(Gal, AutS) \hookrightarrow H^1(Gal, AutgrS). \tag{3}$$

The infinite-dimensional linearly compact simple Lie superalgebras have been classified in [10]. The list consists of ten series ($m \geq 1$): $W(m,n)$, $S(m,n)$ ($(m,n) \neq (1,1)$), $H(m,n)$ ($m$ even), $K(m,n)$ ($m$ odd), $HO(m,m)$ ($m \geq 2$), $SHO(m,m)$ ($m \geq 3$), $KO(m,m+1)$, $SKO(m,m+1;\beta)$ ($m \geq 2$), $SHO^{\sim}(m,m)$ ($m$ even), $SKO^{\sim}(m,m+1)$ ($m \geq 3$ odd), and five exceptional Lie superalgebras: $E(1,6)$, $E(3,6)$, $E(3,8)$, $E(4,4)$, $E(5,10)$. Since the following isomorphisms hold (see [2], [10]): $W(1,1) \cong K(1,2) \cong KO(1,2)$, $S(2,1) \cong HO(2,2) \cong SKO(2,3;0)$, $SHO^{\sim}(2,2) \cong H(2,1)$, when dealing with $W(m,n)$, $KO(n,n+1)$, $HO(n,n)$, $SKO(2,3;\beta)$ and $SHO^{\sim}(n,n)$, we will assume that $(m,n) \neq (1,1)$, $n \geq 2$, $n \geq 3$, $\beta \neq 0$, and $n > 2$, respectively. We will use the construction of all these superalgebras as given in [2] (see also [1], [4], [10], [12], [16]).

Since the first Galois cohomology with coefficients in the groups $GL_n(\bar{F})$ and $Sp_n(\bar{\mathbb{F}})$ is trivial (see, e.g., [14], [15]), we conclude from (3) and Table 1, that

$H^1(Gal, AutS)$ is trivial in all cases except for four: $S = H(m,n)$, $K(m,n)$, $S(1,2)$, and $E(1,6)$. Thus, in all cases, except for these four, $S$ has a unique $\mathbb{F}$-form (in the $SKO(n,n+1;\beta)$ case we have to assume that $\beta \in \mathbb{F}$ in order for such a form to exist).

Since $H^1(Gal, O_n(\bar{\mathbb{F}}))$ is in canonical bijective correspondence with classes of non-degenerate bilinear forms in $n$ variables over $\mathbb{F}$ (see, e.g., [15]), we find that all $\mathbb{F}$-forms of $H(m,n)$ and $K(m,n)$ are defined by the action on supersymplectic and supercontact forms over $\mathbb{F}$, respectively. In the cases $S = S(1,2)$ and $E(1,6)$, the answer is more interesting. We construct all $\mathbb{F}$-forms of these Lie superalgebras, using the theory of Lie conformal superalgebras.

The present paper is a continuation of [2], which we refer to for terminology not explained here. The base field, unlike in [2], is an arbitrary field $\mathbb{F}$ of characteristic 0, and we denote by $\bar{\mathbb{F}}$ its algebraic closure.

In the Lie algebra case the problems considered in the present paper were solved by Rudakov [13], whose methods we use.

# 1 $\mathbb{Z}$-Gradings

In papers [2] and [10] the base field is $\mathbb{C}$. However, it is not difficult to extend all the results there to the case of an arbitrary algebraically closed field $\bar{\mathbb{F}}$ of characteristic zero. In order to do this one has to replace exponentiable derivations of a linearly compact algebra $S$ in the sense of [2], [10], by exponentiable derivations in the sense of [8] (a derivation $d$ of a Lie superalgebra $S$ over a field $\mathbb{F}$ is called *exponentiable* in the sense of [8] if $d(H) \subset H$ for any closed $AutS$-invariant subspace $H$ of $S$). Also, we define the *group of inner automorphisms* of $S$ to be the group generated by all elements $\exp(ad\ a)$, where $\exp(ad\ a)$ converges in linearly compact topology. Then Theorem 1.7 of [2] on conjugacy of maximal tori in an artinian semisimple linearly compact superalgebra still holds over $\bar{\mathbb{F}}$. Consequently, the classification given in [2] of primitive pairs $(L, L_0)$ up to conjugacy by inner automorphisms of $DerL$ stands as well over $\bar{\mathbb{F}}$.

We first recall from [2] and [10] the necessary information on $\mathbb{Z}$-gradings of Lie superalgebras in question over the field $\bar{\mathbb{F}}$. For information on finite-dimensional Lie superalgebras we refer to [9] or [10].

Recall that $W(m,n)$ is the Lie superalgebra of all continuous derivations of the commutative associative superalgebra $\Lambda(m,n) = \Lambda(n)[[x_1,\dots,x_m]]$, where $\Lambda(n)$ is the Grassmann superalgebra in $n$ odd indeterminates $\xi_1,\dots,\xi_n$, and $x_1,\dots,x_m$ are even indeterminates. Recall that a $\mathbb{Z}$-grading of the Lie superalgebra $W(m,n)$ is called the grading of type $(a_1,\dots,a_m|b_1,\dots,b_n)$ if $a_i = \deg x_i = -\deg \frac{\partial}{\partial x_i} \in \mathbb{N}$ and $b_i = \deg \xi_i = -\deg \frac{\partial}{\partial \xi_i} \in \mathbb{Z}$ (cf. [10, Example 4.1]). Every such a grading always induces a grading on the Lie superalgebra $S(m,n)$ and it induces a grading on $S = H(m,n)$, $K(m,n)$, $HO(n,n)$, $SHO(n,n)$, $KO(n,n+1)$, or $SKO(n,n+1;\beta)$ if the defining differential form of $S$ is homogeneous with respect to this grading. The induced grading on $S$ is also called a grading of type $(a_1,\dots,a_m|b_1,\dots,b_n)$.

The $\mathbb{Z}$-grading of type $(1,\dots,1|1,\dots,1)$ is an irreducible grading of $W(m,n)$

called its *principal* grading. In this grading $W(m,n) = \prod_{j\geq -1} \mathfrak{g}_j$ has 0-th graded component isomorphic to the Lie superalgebra $gl(m,n)$ and $-1$-st graded component isomorphic to the standard $gl(m,n)$-module $\bar{\mathbb{F}}^{m|n}$. The even part of $\mathfrak{g}_0$ is isomorphic to the Lie algebra $gl_m \oplus gl_n$ where $gl_m$ (resp. $gl_n$) acts trivially on $\bar{\mathbb{F}}^n$ (resp. $\bar{\mathbb{F}}^m$) and acts as the standard representation on $\bar{\mathbb{F}}^m$ (resp. $\bar{\mathbb{F}}^n$).

The principal grading of $W(m,n)$ induces on $S(m,n)$, $H(m,n)$, $HO(n,n)$ and $SHO(n,n)$, irreducible gradings also called *principal*.

The 0-th graded component of $S(m,n)$ in its principal grading is isomorphic to the Lie superalgebra $sl(m,n)$ and its $-1$-st graded component is isomorphic to the standard $sl(m,n)$-module $\bar{\mathbb{F}}^{m|n}$. The even part of $\mathfrak{g}_0$ is isomorphic to the Lie algebra $sl_m \oplus sl_n \oplus \bar{\mathbb{F}}c$ where $sl_m$ (resp. $sl_n$) acts trivially on $\bar{\mathbb{F}}^n$ (resp. $\bar{\mathbb{F}}^m$) and acts as the standard representation on $\bar{\mathbb{F}}^m$ (resp. $\bar{\mathbb{F}}^n$). Here $c$ acts by multiplication by $-n$ (resp. $-m$) on $\bar{\mathbb{F}}^m$ (resp. $\bar{\mathbb{F}}^n$).

Let $S = H(2k,n) = \prod_{j\geq -1} \mathfrak{g}_j$ with its principal grading. Recall that the Lie superalgebra $H(2k,n)$ can be identified with $\Lambda(2k,n)/\bar{\mathbb{F}}1$, where we have $2k$ even indeterminates $q_1,\dots,q_k$, $p_1,\dots,p_k$, and $n$ odd indeterminates $\xi_1,\dots,\xi_n$, with bracket $[f,g] = \sum_{i=1}^k (\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}) - (-1)^{p(f)} \sum_{i=1}^n \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_{n-i+1}}$. Then $\mathfrak{g}_0 \cong spo(2k,n)$, and $\mathfrak{g}_{-1}$ is isomorphic to the standard $spo(2k,n)$-module $\bar{\mathbb{F}}^{2k|n}$. Here $(\mathfrak{g}_0)_{\bar{0}}$ is spanned by elements $\{p_ip_j, p_iq_j, q_iq_j\}$ for $i,j = 1,\dots,k$, and $\{\xi_i\xi_j\}_{i\neq j}$ for $i,j = 1,\dots,n$, hence it is isomorphic to $sp_{2k} \oplus so_n$. The odd part of $\mathfrak{g}_0$ is spanned by vectors $\{p_i\xi_j, q_i\xi_j\}$ for $i = 1,\dots,k$ and $j = 1,\dots,n$, hence it is isomorphic to the $sp_{2k} \oplus so_n$-module $\bar{\mathbb{F}}^{2k} \otimes \bar{\mathbb{F}}^n$, where $\bar{\mathbb{F}}^{2k}$ and $\bar{\mathbb{F}}^n$ are the standard $sp_{2k}$ and $so_n$-modules, respectively. Besides, $\mathfrak{g}_{-1} = \langle p_i, q_i, \xi_j \mid i = 1,\dots,k, j = 1,\dots,n\rangle$, hence $sp_{2k}$ acts trivially on $\bar{\mathbb{F}}^n = \langle \xi_j \mid j = 1,\dots,n\rangle$ and acts as the standard representation on $\bar{\mathbb{F}}^{2k} = \langle p_i, q_i \mid i = 1,\dots,k\rangle$, and $so_n$ acts trivially on $\bar{\mathbb{F}}^{2k}$ and by the standard action on $\bar{\mathbb{F}}^n$.

The grading of type $(2,1,\dots,1|1,\dots,1)$ of $W(2k+1,n)$ induces an irreducible grading $K(2k+1,n) = \prod_{j\geq -2} \mathfrak{g}_j$, called the *principal* grading of $K(2k+1,n)$. Its 0-th graded component $\mathfrak{g}_0$ is isomorphic to the Lie superalgebra $cspo(2k,n)$ and $\mathfrak{g}_{-1}$ is isomorphic to the standard $cspo(2k,n)$-module $\bar{\mathbb{F}}^{2k|n}$.

Consider the Lie superalgebra $S = HO(n,n) = \prod_{j\geq -1} \mathfrak{g}_j$ with its principal grading. Then $\mathfrak{g}_0$ is isomorphic to the Lie superalgebra $\tilde{P}(n) = \tilde{P}(n)_{-1} + \tilde{P}(n)_0 + \tilde{P}(n)_1$, where $\tilde{P}(n)_0 \cong gl_n$, and, as $gl_n$-modules, $\tilde{P}(n)_{-1} \cong \Lambda^2(\bar{\mathbb{F}}^{n*})$, $\tilde{P}(n)_1 \cong S^2(\bar{\mathbb{F}}^n)$, and $\mathfrak{g}_{-1} \cong \bar{\mathbb{F}}^n \oplus \bar{\mathbb{F}}^{n*}$, where $\bar{\mathbb{F}}^n$ is the standard $gl_n$-module, and the $gl_n$-submodules $\bar{\mathbb{F}}^n$ and $\bar{\mathbb{F}}^{n*}$ of $\mathfrak{g}_{-1}$ have different parities.

The 0-th graded component of $SHO(n,n)$ in its principal grading is isomorphic to the graded subalgebra $P(n) = P(n)_{-1} + P(n)_0 + P(n)_1$ of $\tilde{P}(n)$, where $P(n)_0 \cong sl_n$, $P(n)_{-1} \cong \Lambda^2(\bar{\mathbb{F}}^{n*})$ and $P(n)_1 \cong S^2(\bar{\mathbb{F}}^n)$, and its $-1$-st graded component is isomorphic to the standard $P(n)$-module $\bar{\mathbb{F}}^n \oplus \bar{\mathbb{F}}^{n*}$.

The $\mathbb{Z}$-grading of type $(1,\dots,1|1,\dots,1,2)$ of $W(n,n+1)$ induces on the Lie superalgebras $KO(n,n+1)$ and $SKO(n,n+1;\beta)$ an irreducible grading called *principal*. In these cases the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is obtained from that of $SHO(n,n)$ by adding some operators which act as scalars on $\bar{\mathbb{F}}^n$ and $\bar{\mathbb{F}}^{n*}$.

The $\mathbb{Z}$-grading of type $(1,\dots,1|0,\dots,0)$ of $S = W(m,n)$, $S(m,n)$, $HO(n,n)$,

$SHO(n,n)$, the $\mathbb{Z}$-gradings of type $(1,\ldots,1|2,\ldots,2,0,\ldots,0)$ and $(2,1,\ldots,1|2,\ldots,2,0,\ldots,0)$ with $h$ zeros of $S=H(m,2h)$ and $K(m,2h)$, respectively, and the $\mathbb{Z}$-grading of type $(1,\ldots,1|0,\ldots,0,1)$ of $S=KO(n,n{+}1)$, $SKO(n,n{+}1;\beta)$, is called the *subprincipal* grading of $S$.

The Lie superalgebra $S=SKO(2,3;1)=\prod_{j\geq -1}\mathfrak{g}_j$ in its subprincipal grading has 0-th graded component $\mathfrak{g}_0$ isomorphic to the semidirect sum of $S(0,2)$ and the subspace of $\Lambda(2)$ spanned by all the monomials except for $\xi_1\xi_2$, and $\mathfrak{g}_{-1}\cong\Lambda(2)$. The even part of $\mathfrak{g}_0$ is isomorphic to $sl_2\oplus\bar{\mathbb{F}}$ and, as an $sl_2$-module, $\mathfrak{g}_{-1}=\bar{\mathbb{F}}^2\oplus\bar{\mathbb{F}}^2$, where the two copies of $\bar{\mathbb{F}}^2$ have different parities and $sl_2$ acts by the standard action on the even copy and trivially on the odd copy. The algebra of outer derivations of $S$ is isomorphic to $sl_2$ (cf. [2, Remark 4.15]); it acts trivially on the even subspace of $\mathfrak{g}_{-1}$ and by the standard action on the odd one. Finally, $\bar{\mathbb{F}}$ acts on $\mathfrak{g}_{-1}$ by multiplication by $-2$.

The $\mathbb{Z}$-grading of $W(1,2)$ of type $(2|1,1)$ induces a grading on $S(1,2)=\prod_{j\geq -2}\mathfrak{g}_j$, which is not irreducible. Then $\mathfrak{g}_0\cong sl_2\oplus\bar{\mathbb{F}}c$ where $c$ acts on $S$ as the grading operator, and $\mathfrak{g}_{-1}=\bar{\mathbb{F}}^2\oplus\bar{\mathbb{F}}^2$, where $\bar{\mathbb{F}}^2$ is the standard $sl_2$-module. The two copies of the standard $sl_2$-module in $\mathfrak{g}_{-1}$ are both odd.

Likewise, the $\mathbb{Z}$-grading of $W(3,3)$ of type $(2,2,2|1,1,1)$ induces a grading on $SHO(3,3)=\prod_{j\geq -2}\mathfrak{g}_j$ , which is not irreducible. Here $\mathfrak{g}_0\cong sl_3$ and $\mathfrak{g}_{-1}=\bar{\mathbb{F}}^3\oplus\bar{\mathbb{F}}^3$, where $\bar{\mathbb{F}}^3$ is the standard $sl_3$-module. The two copies of the standard $sl_3$-module in $\mathfrak{g}_{-1}$ are both odd.

Consider the Lie superalgebra $K(1,6)=\prod_{j\geq -2}\mathfrak{g}_j$ with its principal grading. Then $\mathfrak{g}_0=sl_4\oplus\bar{\mathbb{F}}c$ and $\mathfrak{g}_{-1}\cong\Lambda^2\bar{\mathbb{F}}^4$, where $\bar{\mathbb{F}}^4$ denotes the standard $sl_4$-module, $\mathfrak{g}_1\cong\mathfrak{g}_{-1}^*\oplus\mathfrak{g}_1^+\oplus\mathfrak{g}_1^-$, as $sl_4$-modules, with $\mathfrak{g}_1^+\cong S^2\bar{\mathbb{F}}^4$ and $\mathfrak{g}_1^-\cong S^2(\bar{\mathbb{F}}^{4^*})$. The Lie superalgebra $S=E(1,6)$ is the graded subalgebra of $K(1,6)$ generated by $\mathfrak{g}_{-1}+\mathfrak{g}_0+(\mathfrak{g}_{-1}^*+\mathfrak{g}_1^+)$ (cf. [10, Example 5.2], [4, §4.2], [16, §3]). It follows that the $\mathbb{Z}$-grading of type $(2|1,1,1,1,1,1)$ induces on $E(1,6)$ an irreducible grading, called the *principal* grading of $E(1,6)$, where $\mathfrak{g}_{-1}=\langle\xi_i,\eta_i\rangle$, $\mathfrak{g}_{-1}^*=\langle t\xi_i,t\eta_i\rangle$ and $\mathfrak{g}_1^+=\langle\xi_1\xi_2\xi_3,\xi_1\eta_2\eta_3,\xi_2\eta_1\eta_3,\xi_3\eta_1\eta_2,\xi_1(\xi_2\eta_2+\xi_3\eta_3),\xi_2(\xi_1\eta_1+\xi_3\eta_3),\eta_3(\xi_1\eta_1-\xi_2\eta_2),\xi_3(\xi_1\eta_1+\xi_2\eta_2),\eta_2(\xi_1\eta_1-\xi_3\eta_3),\eta_1(\xi_2\eta_2-\xi_3\eta_3)\rangle$, and $\mathfrak{g}_1^-$ is obtained from $\mathfrak{g}_1^+$ by exchanging $\xi_i$ with $\eta_i$ for every $i=1,2,3$.

Next, the *principal* grading of $E(3,6)$ is an irreducible grading of depth two whose 0-th graded component is isomorphic to $sl_3\oplus sl_2\oplus\mathbb{C}c$, and whose $-1$-st graded component is isomorphic, as an $sl_3\oplus sl_2$-module, to $\bar{\mathbb{F}}^3\boxtimes\bar{\mathbb{F}}^2$ where $\bar{\mathbb{F}}^3$ and $\bar{\mathbb{F}}^2$ denote the standard $sl_3$ and $sl_2$-modules, respectively. Here $c$ acts on $E(3,6)$ as the grading operator (with respect to its principal grading). Likewise, the *principal* grading of $E(3,8)$ is an irreducible grading of depth three whose 0-th graded component is isomorphic to $sl_3\oplus sl_2\oplus\mathbb{C}c$ , and whose $-1$-st graded component is isomorphic, as an $sl_3\oplus sl_2$-module, to $\bar{\mathbb{F}}^3\boxtimes\bar{\mathbb{F}}^2$ where $\bar{\mathbb{F}}^3$ and $\bar{\mathbb{F}}^2$ denote the standard $sl_3$ and $sl_2$-modules, respectively; $c$ acts on $E(3,8)$ as the grading operator.

The Lie superalgebra $S=E(4,4)$ has even part isomorphic to $W_4$ and odd part isomorphic to the $W_4$-module $\Omega^1(4)^{-\frac{1}{2}}$. The bracket between two odd elements $\omega_1$ and $\omega_2$ is defined as: $[\omega_1,\omega_2]=d\omega_1\wedge\omega_2+\omega_1\wedge d\omega_2$. The *principal* grading of $S$ is an irreducible $\mathbb{Z}$-grading of depth 1 whose 0-th graded

component $\mathfrak{g}_0 = \langle x_i \frac{\partial}{\partial x_j}, x_i dx_j \rangle$ is isomorphic to the Lie superalgebra $\hat{P}(4)$ and $\mathfrak{g}_{-1} = \langle \frac{\partial}{\partial x_j}, dx_j \rangle$ is isomorphic to the standard $\hat{P}(4)$-module $\bar{\mathbb{F}}^{4|4}$. We recall that $\hat{P}(4) = P(4) + \bar{\mathbb{F}}z$ is a (non-trivial) central extension of $P(4)$ with center $\bar{\mathbb{F}}z$ (see [2], [10], [16]).

Finally, the *principal* grading of the Lie superalgebra $E(5,10)$ is irreducible of depth 2, with 0-th graded component isomorphic to $sl_5$ and $-1$-st graded component isomorphic ro $\Lambda^2\bar{\mathbb{F}}^5$, where $\bar{\mathbb{F}}^5$ is the standard $sl_5$-module.

Given a simple infinite-dimensional linearly compact Lie superalgebra $S = \prod_{j\geq -d} \mathfrak{g}_j$ with its principal or subprincipal grading, we will call $S_0 = \prod_{j\geq 0} \mathfrak{g}_j$ the principal or subprincipal subalgebra of $S$, respectively. Likewise, if $S = \prod_{j\geq -d} \mathfrak{g}_j$ with a grading of a given type, we will call $S_0 = \prod_{j\geq 0} \mathfrak{g}_j$ the subalgebra of $S$ of this type.

**Remark 1.1** One can show that every non-graded maximal open subalgebra of any non-exceptional simple infinite-dimensional linearly compact Lie superalgebra $S$ in its defining embedding in $W(m,n)$, can be constructed as the intersection of $S$ with a graded subalgebra of $W(m,n)$. For example, the maximal open subalgebra $L_0(0)$ of $S = H(m,1)$ constructed in [2, Example 3.3], is the intersection of $S$ with the subprincipal subalgebra of $W(m,1)$. Since the supersymplectic form is not homogeneous with respect to the subprincipal grading of $W(m,1)$, $L_0(0)$ is not graded. We shall call this subalgebra the *subprincipal* subalgebra of $H(m,1)$.

If $\mathfrak{g}$ is a Lie algebra acting linearly on a vector space $V$ over $\bar{\mathbb{F}}$, we denote by $\exp(\mathfrak{g})$ the linear algebraic subgroup of $GL(V)$, generated by all $\exp a$, where $a$ is a (locally) nilpotent endomorphism of $V$, and by $t^a$, where $a$ is a diagonalizable endomorphism of $V$ with integer eigenvalues and $t \in \bar{\mathbb{F}}^\times$.

If a group $G$ is an almost direct product of two subgroups $G_1$ and $G_2$ (i.e., both $G_1$ and $G_2$ are normal subgroups and $G_1 \cap G_2$ is a finite central subgroup of $G$) we will denote it by $G = G_1 \cdot G_2$. We will often make use of the following simple result:

**Proposition 1.2** *Suppose we have a representation of a Lie superalgebra* $\mathfrak{g}$ *over* $\bar{\mathbb{F}}$ *in a vector superspace* $V$*, and a faithful representation of a group* $G$ *in* $V$*, containing* $\exp(\mathfrak{g}_{\bar{0}})$*, preserving parity and such that conjugation by elements of* $G$ *induces automorphisms of* $\mathfrak{g}$*. Then the maximal possible* $G$ *are as follows in the following cases:*

(*a*) *if* $\mathfrak{g} = sl_n$ *and* $V = \bar{\mathbb{F}}^n \oplus \bar{\mathbb{F}}^n$ *with the same parity, then* $G$ *is an almost direct product of* $GL_2$ *and* $SL_n$*; in particular if* $n = 2$ *then* $G = \bar{\mathbb{F}}^\times \cdot SO_4$ *and if* $n = 3$ *then* $G = \bar{\mathbb{F}}^\times \cdot (SL_2 \times SL_3)$*;*

(*b*) *if* $\mathfrak{g} = sl_2 \oplus sl_3$ *and* $V = \bar{\mathbb{F}}^3 \boxtimes \bar{\mathbb{F}}^2$*, then* $G = \bar{\mathbb{F}}^\times \cdot (SL_2 \times SL_3)$*;*

(*c*) *if* $\mathfrak{g} = sl_5$ *and* $V = \Lambda^2\bar{\mathbb{F}}^5$*, then* $G = GL_5$*;*

(*d*) *if* $\mathfrak{g} = sl(m,n)$ *and* $V = \bar{\mathbb{F}}^{m|n}$ *is the standard* $sl(m,n)$*-module, then* $G = GL_m \times GL_n$*;*

(e) *if* $\mathfrak{g} = spo(2k, n)$ *and* $V = \bar{\mathbb{F}}^{2k|n}$ *is the standard* $spo(2k, n)$*-module, then* $G = \bar{\mathbb{F}}^{\times} \cdot (Sp_{2k} \times O_n)$*;*

(f) *if* $\mathfrak{g} = P_n$ *and* $V = \bar{\mathbb{F}}^n + \bar{\mathbb{F}}^{n*}$ *is the standard* $P_n$*-module, then* $G = \bar{\mathbb{F}}^{\times} \cdot GL_n$*;*

(g) *if* $\mathfrak{g} = \hat{P}_4$ *and* $V = \bar{\mathbb{F}}^4 + \bar{\mathbb{F}}^{4*}$ *is the standard* $\hat{P}_4$*-module, then* $G = GL_4$*.*

*In all cases when* $G = \bar{\mathbb{F}}^{\times} \cdot G_1$*, the group* $\bar{\mathbb{F}}^{\times}$ *acts on* $V$ *by scalar multiplication.*

**Proof.** Consider the map $f : G \longrightarrow Aut(\mathfrak{g})$ that associates to every element of $G$ the induced automorphism by conjugation of $\mathfrak{g}$. Then the kernel of $f$ consists of the elements of $G$ commuting with $\mathfrak{g}$. Suppose, as in $(a)$, that $\mathfrak{g} = sl_n$ and $V = \bar{\mathbb{F}}^n \oplus \bar{\mathbb{F}}^n$, where the two copies of the standard $sl_n$-module have the same parity. Then $Imf$ consists of inner automorphisms of $sl_n$, i.e., $Imf = PGL_n$, and $\ker f = GL_2$. We have therefore the following exact sequence:

$$1 \longrightarrow GL_2 \longrightarrow G \longrightarrow PGL_n \longrightarrow 1.$$

Since there is in $G$ a complementary to $GL_2$ subgroup, which is $SL_n$, we conclude that $G$ is an almost direct product of $GL_2$ and $SL_n$. It follows that if $n$ is odd, then $G = \bar{\mathbb{F}}^{\times} \cdot (SL_2 \times SL_n)$, and if $n$ is even then $G = \bar{\mathbb{F}}^{\times} \cdot ((SL_2 \times SL_n)/C_2)$ where $C_2$ is the cyclic subgroup of order two of $SL_2 \times SL_n$ generated by $(-I_2, -I_n)$, proving $(a)$. The same argument proves $(b)$.

By the same argument, in case $(c)$ we get the exact sequence

$$1 \longrightarrow \bar{\mathbb{F}}^{\times} \longrightarrow G \longrightarrow PGL_5 \longrightarrow 1.$$

Since $G$ contains a complementary to $\bar{\mathbb{F}}^{\times}$ subgroup, which is $SL_5$, we conclude that $G = GL_5$.

If $\mathfrak{g} = sl(m, n)$ and $V = \bar{\mathbb{F}}^{m|n}$ is its standard representation, then $gl_m$ acts irreducibly on $\bar{\mathbb{F}}^m$ and $gl_n$ acts irreducibly on $\bar{\mathbb{F}}^n$, hence $G = GL_m \times GL_n$, proving $(d)$.

Suppose that $\mathfrak{g} = spo(2k, n)$ and $V = \bar{\mathbb{F}}^{2k|n}$ is the standard $spo(2k, n)$-module. Define on $\mathfrak{g}_{-1} = \langle p_i, q_i, \xi_j \mid i = 1, \dots, k,\ j = 1, \dots, n\rangle$ the following symmetric bilinear form: $(p_i, q_j) = \delta_{i,j}$, $(\xi_i, \xi_j) = \delta_{i,n-j+1}$, $(p_i, p_j) = 0 = (q_i, q_j) = (p_i, \xi_j) = (q_i, \xi_j)$. Then $G$ consists of the automorphisms of $\bar{\mathbb{F}}^{2k} \oplus \bar{\mathbb{F}}^n$ preserving the bilinear form $(\cdot, \cdot)$ up to multiplication by a scalar, hence $G = \bar{\mathbb{F}}^{\times} \cdot (Sp_{2k} \times O_n)$, proving $(e)$.

Finally, let $\mathfrak{g} = P_n$ and $V = \bar{\mathbb{F}}^n + \bar{\mathbb{F}}^{n*}$ be the standard $P_n$-module. Then $gl_n$ acts irreducibly on $\bar{\mathbb{F}}^n$ and $\bar{\mathbb{F}}^{n*}$ which have different parities. It follows that $G = \bar{\mathbb{F}}^{\times} \cdot GL_n$. Likewise, if $\mathfrak{g} = \hat{P}(4)$, then $G = \bar{\mathbb{F}}^{\times} \cdot SL_4$ since the group of automorphisms of $\mathfrak{g}$ is $SL_4$. $\square$

# 2 On *AutS*

Let $S$ be a linearly compact infinite-dimensional Lie superalgebra over $\bar{\mathbb{F}}$ and let $AutS$ denote the group of all continous automorphisms of $S$. Let $S = S_{-d} \supset$

$\dots \supset S_0 \supset \dots$ be a filtration of $S$ by open subalgebras such that all $S_j$ are $AutS$-invariant and $GrS = \oplus_{j \geq -d} \mathfrak{g}_j$ is a transitive graded Lie superalgebra. Denote by $Aut(GrS)$ the group of automorphisms of $GrS$ preserving the grading. Denote by $AutfS$ the subgroup consisting of $g \in AutS$ which induce an identity automorphism of $GrS$, and let $AutgrS$ be the subgroup of $Aut(GrS)$ consisting of automorphisms induced by $g \in AutS$. We have an exact sequence:

$$1 \to AutfS \to AutS \to AutgrS \to 1. \tag{4}$$

**Proposition 2.1** *(a) The restriction map $AutgrS \to GL(\mathfrak{g}_{-1})$ is injective.*
*(b) $AutfS$ consists of inner automorphisms of $DerS$. In fact $AutfS = \exp ad\,(DerS)_1$, where $(DerS)_1$ is the first member of the filtration of $DerS$, induced by that of $S$.*
*(c) If $S_0$ is a graded subalgebra, i.e., $S_j = \mathfrak{g}_j \oplus S_{j+1}$ for all $j \geq -d$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, then $AutgrS = Aut(GrS)$ and*

$$AutS = AutfS \rtimes Aut(GrS).$$

**Proof.** By transitivity, $\mathfrak{g}_{-n} = \mathfrak{g}_{-1}^n$ for $n \geq 1$, and, in addition, we have the well-known injective $AutgrS$-equivariant map $\mathfrak{g}_n \to Hom(\mathfrak{g}_{-1}^{\otimes(n+1)}, \mathfrak{g}_{-1})$ for $n \geq 0$, which implies $(a)$.

If $\sigma \in AutfS$, then $\sigma = 1 + \sigma_1$, where $\sigma_1(L_j) \subset L_{j+1}$. Hence $\log \sigma = \sum_{n \geq 1} (-1)^{n+1} \frac{\sigma_1^n}{n}$ converges and $e^{t \log \sigma}$ converges to a one-parameter subgroup of $AutfS$. Hence $\sigma$ is an inner automorphism of $DerS$, proving $(b)$.

If $S_0$ is a graded subalgebra, we have an obvious inclusion $Aut(GrS) \subset AutS$ and exact sequence (4), proving $(c)$. $\square$

**Remark 2.2** By Proposition 2.1(a), $AutgrS$ is a subgroup of $GL(\mathfrak{g}_{-1})$ whose Lie algebra is $Gr_0 DerS$ acting on $\mathfrak{g}_{-1}$. By Proposition 2.1(b), $AutfS$ is a prounipotent group.

# 3 Invariant Subalgebras

Given a linearly compact Lie superalgebra $L$, we call *invariant* a subalgebra of $L$ which is invariant with respect to all its inner automorphisms, or, equivalently, which contains all elements $a$ of $L$ such that $\exp(ad(a))$ converges in the linearly compact topology. It turns out that an open subalgebra of minimal codimension in a linearly compact infinite-dimensional simple Lie superalgebra $S$ over $\bar{\mathbb{F}}$ is always invariant under all inner automorphisms of $S$ (see [2]).

**Example 3.1** We recall that the Lie superalgebra $S = SHO^{\sim}(n, n)$ is the subalgebra of $HO(n, n)$ defined as follows:

$$SHO^{\sim}(n, n) = \{X \in HO(n, n) \mid X(Fv) = 0\}$$

where $v$ is the volume form associated to the usual divergence and $F = 1 - 2\xi_1 \dots \xi_n$ (cf. [2, §5]). Let $S_0$ be the intersection of $S$ with the principal subalgebra of $W(n,n)$. Then the Weisfeiler filtration associated to $S_0$ has depth one and $\overline{GrS} \cong SHO'(n,n)$ with the $\mathbb{Z}$-grading of type $(1,\dots,1|1,\dots,1)$ (cf. [2, Example 5.2]). Here and further by $\overline{GrS}$ we denote the completion of the graded Lie superalgebra associated to the above filtration. By [2, Proposition 1.11], $S_0$ is a maximal open subalgebra of $S$. It is easy to see that it is also an invariant subalgebra. This subalgebra is called the *principal* subalgebra of $S$.

**Example 3.2** We recall that the Lie superalgebra $S = SKO^{\sim}(n,n+1)$ is the subalgebra of $KO(n,n+1)$ defined as follows:

$$SKO^{\sim}(n,n+1) = \{X \in KO(n,n+1) \mid X(Fv_\beta) = 0\}$$

where $v_\beta$ is the volume form attached to the divergence $div_\beta$ for $\beta = (n+2)/n$ and $F = 1 + \xi_1 \dots \xi_{n+1}$ (cf. [2, §5]). Let $S_0$ be the intersection of $S$ with the subalgebra of $W(n,n+1)$ of type $(1,\dots,1|1,\dots,1,2)$. Then the Weisfeiler filtration associated to $S_0$ has depth 2 and $\overline{GrS} \cong SKO(n,n+1;(n+2)/n)$ with its principal grading. By [2, Proposition 1.11], $S_0$ is a maximal open subalgebra of $S$. It is easy to see that it is also an invariant subalgebra. This subalgebra is called the *principal* subalgebra of $S$.

A complete list of invariant maximal open subalgebras in all simple linearly compact infinite-dimensional Lie superalgebras over $\bar{\mathbb{F}}$, is given in the following theorem (cf. [2, Theorem 11.1]):

**Theorem 3.3** *The following is a complete list of invariant maximal open subalgebras in infinite-dimensional linearly compact simple Lie superalgebras $S$ over $\bar{\mathbb{F}}$:*

(a) *the principal subalgebra of $S$;*

(b) *the subprincipal subalgebra of $S = W(m,1)$, $S(m,1)$, $H(m,1)$, $H(m,2)$, $K(m,2)$, $KO(2,3)$, $SKO(2,3;\beta)$, the subalgebra of type $(2,1,\dots,1|0,2)$ of $K(m,2)$ and the subalgebra of type $(1,\dots,1|0,2)$ of $H(m,2)$;*

(c) *the subalgebra of type $(1,1|-1,-1,0)$ of $SKO(2,3;\beta)$ for $\beta \neq 1$;*

(d) *the subalgebras of $S = S(1,2)$, $SHO(3,3)$, $SKO(2,3;1)$ conjugate to the principal subalgebra by the subgroup of $AutS$ generated by the automorphisms $\exp(ad(\mathfrak{a}))$ where $\mathfrak{a}$ is the algebra of outer derivations of $S$;*

(e) *the subalgebras of $S = SKO(3,4;1/3)$ conjugate to the subprincipal subalgebra by the automorphisms $\exp(ad(t\xi_1\xi_2\xi_3))$ with $t \in \bar{\mathbb{F}}$.*

**Theorem 3.4** *The following is a complete list of maximal among the open $AutS$-invariant subalgebras in infinite-dimensional linearly compact simple Lie superalgebras $S$:*

($a$) *the principal subalgebra of* $S \neq S(1,2)$*,* $SHO(3,3)$*, and* $SKO(2,3;1)$*;*

($b$) *the subprincipal subalgebra of* $S = W(m,1)$*,* $S(m,1)$*,* $H(m,1)$*,* $KO(2,3)$*, and* $SKO(2,3;\beta)$*;*

($c$) *the subalgebra of type* $(1,1|-1,-1,0)$ *in* $S = SKO(2,3;\beta)$ *with* $\beta \neq 1$*;*

($d$) *the subalgebras of type* $(2|1,1)$ *and* $(2,2,2|1,1,1)$ *in* $S = S(1,2)$ *and* $SHO(3,3)$*, respectively.*

**Proof.** We will prove that the subalgebras listed in $(a)-(d)$ are $AutS$-invariant, and, in order to show that they exhaust all maximal among open $AutS$-invariant subalgebras of $S$, it suffices to show that for every subalgebra $S'_0$ of $S$ listed in Theorem 3.3, $\cap_{\varphi\in AutS}\varphi(S'_0)$ is contained in one of them. Indeed, if $S_0$ is a maximal among the $AutS$-invariant open subalgebras of $S$, then $S_0$ is an invariant subalgebra of $S$, hence, every maximal open subalgebra $S'_0$ of $S$ containing $S_0$, is invariant, i.e., $S'_0$ is one of the subalgebras of $S$ listed in Theorem 3.3. Therefore, $S_0 \subset \cap_{\varphi\in AutS}\varphi(S'_0)$.

If $S \neq W(m,1)$, $S(m,1)$, $H(m,1)$, $H(m,2)$, $K(m,2)$, $KO(2,3)$, $SKO(2,3;\beta)$, $S(1,2)$, $SHO(3,3)$ and $SKO(3,4;1/3)$, then, in view of Theorem 3.3, the principal subalgebra of $S$ is the unique invariant maximal open subalgebra of $S$, hence it is invariant with respect to all automorphisms of $S$ and it is the unique maximal among open $AutS$-invariant subalgebras of $S$.

If $S = W(m,1)$, $S(m,1)$, or $KO(2,3)$, then, according to Theorem 3.3, $S$ has two invariant subalgebras: the principal and subprincipal subalgebras. These two subalgebras have different codimension hence, each of them is invariant with respect to all automorphisms of $S$.

If $S = H(m,1)$, then $S$ has two invariant maximal open subalgebras: the principal and the subprincipal subalgebras. Since the principal subalgebra is graded and the subprincipal subalgebra is not (see Remark 1.1), each of them is invariant with respect to all automorphisms of $S$.

If $S = H(m,2)$, then the principal subalgebra is the unique subalgebra of $S$ of minimal codimension, hence it is $AutS$-invariant. Besides, it is the unique maximal among $AutS$-invariant subalgebras of $S$. Indeed, the invariant subalgebras of $S$ of type $(1,\dots,1|2,0)$ and $(1,\dots,1|0,2)$ are conjugate by an outer automorphism of $S$ and their intersection is contained in the principal subalgebra of $S$. By the same arguments, if $S = K(m,2)$, then the principal subalgebra of $S$ is its unique maximal among open $AutS$-invariant subalgebras.

If $S = SKO(3,4;1/3)$, then, according to Theorem 3.3, $S$ has infinitely many invariant subalgebras which are conjugate to the subprincipal subalgebra. Besides, the principal subalgebra of $S$ is an invariant subalgebra. Note that the principal grading of $S$ has depth 2 and the subprincipal grading of $S$ has depth 1, therefore the principal and subprincipal subalgebras are not conjugate. It follows that the principal subalgebra is invariant with respect to all automorphisms of $S$. In fact, it is the unique maximal among $AutS$-invariant subalgebras of $S$, since the intersection of all the subalgebras of $S$ listed in Theorem 3.3($e$) is

the subalgebra of $S$ of type $(2,2,2|1,1,1,3)$, which is contained in the principal subalgebra.

If $S = SKO(2,3;1)$, then $S$ has infinitely many invariant subalgebras which are conjugate to the principal subalgebra, besides, the subprincipal subalgebra is also an invariant subalgebra of $S$. The principal and subprincipal subalgebras have codimension $(2|3)$ and $(2|2)$, respectively, hence they cannot be conjugate. It follows that the subprincipal subalgebra is invariant with respect to all automorphisms of $S$. In fact, it is the unique maximal among $AutS$-invariant subalgebras of $S$, since it contains the intersection of all subalgebras which are conjugate to the principal subalgebra (cf. [2, Remark 4.16]).

If $S = SKO(2,3;\beta)$ for $\beta \neq 0,1$, then, according to Theorem 3.3, $S$ has three invariant maximal open subalgebras, i.e., the subalgebras of type $(1,1|1,1,2)$, $(1,1|0,0,1)$ and $(1,1|-1,-1,0)$. The subalgebras of type $(1,1|1,1,2)$ and $(1,1|-1,-1,0)$ have codimension $(2|3)$ and the subalgebra of type $(1,1|0,0,1)$ has codimension $(2|2)$. It follows that the subprincipal subalgebra is invariant with respect to all automorphisms of $S$, since it is the unique subalgebra of minimal codimension. Consider the grading of $S$ of type $(1,1|1,1,2)$: this is an irreducible grading of depth 2, whose 0-th graded component is isomorphic to the Lie superalgebra $\tilde{P}(2) = P(2) + \bar{\mathbb{F}}(\xi_3 + \beta\Phi)$. Its $-2$-nd graded component is $\bar{\mathbb{F}}1$, on which $\xi_3 + \beta\Phi$ acts as the scalar $-2$. Its $-1$-st graded component is spanned by vectors $\{x_i\}$ and $\{\xi_i\}$, with $i = 1,2$, hence it is isomorphic to the standard $P(2)$-module, and $\xi_3 + \beta\Phi$ acts on $\sum_{i=1}^{2} \bar{\mathbb{F}}x_i$ (resp. $\sum_{i=1}^{2} \bar{\mathbb{F}}\xi_i$) as the scalar $-1+\beta$ (resp. $-1-\beta$). Now let us consider the grading of $S$ of type $(1,1|-1,-1,0)$: this is an irreducible grading of depth 2, whose 0-th graded component is isomorphic to the Lie superalgebra $\tilde{P}(2) = P(2) + \bar{\mathbb{F}}(\xi_3 + \beta\Phi)$. Its $-2$-nd graded component is $\bar{\mathbb{F}}\xi_1\xi_2$, on which $\xi_3 + \beta\Phi$ acts as the scalar $-2\beta$. Its $-1$-st graded component is spanned by vectors $\{\xi_i(\xi_3 + (2\beta-1)\Phi)\}$ and $\{\xi_i\}$, with $i = 1,2$, hence it is isomorphic to the standard $P(2)$-module, and $\xi_3 + \beta\Phi$ acts on $\sum_{i=1}^{2} \bar{\mathbb{F}}\xi_i(\xi_3 + (2\beta-1)\Phi)$ (resp. $\sum_{i=1}^{2} \bar{\mathbb{F}}\xi_i$) as the scalar $1-\beta$ (resp. $-1-\beta$). Since we assumed $\beta \neq 1$, the two gradings are not isomorphic, hence the subalgebras of type $(1,1|1,1,2)$ and $(1,1|-1,-1,0)$ are not conjugate by any automorphism of $S$. We conclude that they are invariant with respect to all automorphisms of $S$.

Finally, if $S = S(1,2)$ or $S = SHO(3,3)$, then, each of the subalgebras listed in $(d)$ is the intersection of all invariant maximal open subalgebras of $S$, which lie in an $AutS$-orbit, by Theorem 3.3, thus it is the unique maximal among $AutS$-invariant subalgebras of $S$. □

# 4 The Group *AutgrS*

In this section, for every simple infinite-dimensional linearly compact Lie superalgebra $S$ over $\bar{\mathbb{F}}$, we fix the following maximal among $AutS$-invariant open subalgebras of $S$, which we shall denote by $S_0$:

1. the principal subalgebra of $S \neq S(1,2)$, $SHO(3,3)$, $SKO(2,3;1)$;

2. the subalgebra of type $(2|1,1)$ in $S = S(1,2)$;

3. the subalgebra of type $(2,2,2|1,1,1)$ in $S = SHO(3,3)$;

4. the subprincipal subalgebra of $S = SKO(2,3;1)$.

**Remark 4.1** In [2, §11] we introduced the notion of the *canonical* subalgebra of $S$, defined as the intersection of all subalgebras of minimal codimension in $S$. It follows from the definition that the canonical subalgebra of $S$ is an $AutS$-invariant subalgebra. If $S \neq KO(2,3)$, $SKO(2,3;\beta)$ with $\beta \neq 1$, and $S \neq SKO(3,4;1/3)$, then the maximal among $AutS$-invariant subalgebras $S_0$ of $S$ we have chosen is the canonical subalgebra of $S$.

Let $S_{-1}$ be a minimal subspace of $S$, properly containing the subalgebra $S_0$ and invariant with respect to the group $AutS$, and let $S = S_{-d} \supsetneqq S_{-d+1} \supset \cdots \supset S_{-1} \supset S_0 \supset \cdots$ be the associated Weisfeiler filtration of $S$. All members of the Weisfeiler filtration associated to $S_0$ are invariant with respect to the group $AutS$. Let $GrS = \oplus_{j \geq -d} \mathfrak{g}_j$ be the associated $\mathbb{Z}$-graded Lie superalgebra. In this section we will describe the group $AutgrS$ introduced in Section 2, for every $S$. The results are summarized in Table 1 (where by $\Pi V$ we denote $V$ with reversed parity).

| $S$ | $\mathfrak{g}_0$ | $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ | $AutgrS$ |
|---|---|---|---|
| $W(m,n)$, $(m,n) \neq (1,1)$ | $gl(m,n)$ | $\bar{\mathbb{F}}^{m\|n}$ | $GL_m \times GL_n$ |
| $S(1,2)$ | $sl_2 \oplus \bar{\mathbb{F}}$ | $\Pi\bar{\mathbb{F}}^2 \oplus \Pi\bar{\mathbb{F}}^2$ | $\bar{\mathbb{F}}^\times \cdot SO_4$ |
| $S(m,n)$, $(m,n) \neq (1,2)$ | $sl(m,n)$ | $\bar{\mathbb{F}}^{m\|n}$ | $GL_m \times GL_n$ |
| $H(2k,n)$ | $spo(2k,n)$ | $\bar{\mathbb{F}}^{2k\|n}$ | $\bar{\mathbb{F}}^\times \cdot (Sp_{2k} \times O_n)$ |
| $K(2k+1,n)$ | $cspo(2k,n)$ | $\bar{\mathbb{F}}^{2k\|n}$ | $\bar{\mathbb{F}}^\times \cdot (Sp_{2k} \times O_n)$ |
| $HO(n,n)$, $n > 2$ | $\tilde{P}(n)$ | $\bar{\mathbb{F}}^n \oplus \Pi\bar{\mathbb{F}}^{n*}$ | $\bar{\mathbb{F}}^\times \cdot GL_n$ |
| $SHO(3,3)$ | $sl_3$ | $\Pi\bar{\mathbb{F}}^3 \oplus \Pi\bar{\mathbb{F}}^3$ | $\bar{\mathbb{F}}^\times \cdot (SL_3 \times SL_2)$ |
| $SHO(n,n)$, $n > 3$ | $P(n)$ | $\bar{\mathbb{F}}^n \oplus \Pi\bar{\mathbb{F}}^{n*}$ | $\bar{\mathbb{F}}^\times \cdot GL_n$ |
| $KO(n,n+1)$, $n \geq 2$ | $c\tilde{P}(n)$ | $\bar{\mathbb{F}}^n \oplus \Pi\bar{\mathbb{F}}^{n*}$ | $\bar{\mathbb{F}}^\times \cdot GL_n$ |
| $SKO(2,3;1)$ | $\langle 1, \xi_1, \xi_2 \rangle \rtimes S(0,2)$ | $\Lambda(2)$ | $\bar{\mathbb{F}}^\times \cdot (SL_2 \times SL_2)$ |
| $SKO(2,3;\beta)$, $\beta \neq 0,1$ | $\tilde{P}(2)$ | $\bar{\mathbb{F}}^2 \oplus \Pi\bar{\mathbb{F}}^{2*}$ | $\bar{\mathbb{F}}^\times \cdot GL_2$ |
| $SKO(n,n+1;\beta)$, $n > 2$ | $\tilde{P}(n)$ | $\bar{\mathbb{F}}^n \oplus \Pi\bar{\mathbb{F}}^{n*}$ | $\bar{\mathbb{F}}^\times \cdot GL_n$ |
| $SHO^\sim(n,n)$, $n > 2$ | $P(n)$ | $\bar{\mathbb{F}}^n \oplus \Pi\bar{\mathbb{F}}^{n*}$ | $SL_n$ |
| $SKO^\sim(n,n+1)$ | $\tilde{P}(n)$ | $\bar{\mathbb{F}}^n \oplus \Pi\bar{\mathbb{F}}^{n*}$ | $SL_n$ |
| $E(1,6)$ | $so_6 \oplus \bar{\mathbb{F}}$ | $\Pi\bar{\mathbb{F}}^6$ | $\bar{\mathbb{F}}^\times \cdot SO_6$ |
| $E(3,6)$ | $sl_3 \oplus sl_2 \oplus \bar{\mathbb{F}}$ | $\Pi(\bar{\mathbb{F}}^3 \boxtimes \bar{\mathbb{F}}^2)$ | $\bar{\mathbb{F}}^\times \cdot (SL_2 \times SL_3)$ |
| $E(5,10)$ | $sl_5$ | $\Pi(\Lambda^2\bar{\mathbb{F}}^5)$ | $GL_5$ |
| $E(4,4)$ | $\hat{P}(4)$ | $\bar{\mathbb{F}}^{4\|4}$ | $GL_4$ |
| $E(3,8)$ | $sl_3 \oplus sl_2 \oplus \bar{\mathbb{F}}$ | $\Pi(\bar{\mathbb{F}}^3 \boxtimes \bar{\mathbb{F}}^2)$ | $\bar{\mathbb{F}}^\times \cdot (SL_2 \times SL_3)$ |

**Table 1.**

**Theorem 4.2** *Let $S$ be a simple infinite-dimensional linearly compact Lie superalgebra over $\bar{\mathbb{F}}$. Then $AutgrS$ is the algebraic group listed in the last column of Table 1. In all cases when $AutgrS \cong \bar{\mathbb{F}}^{\times} \cdot G_1$, the group $\bar{\mathbb{F}}^{\times}$ acts on $\mathfrak{g}_{-1}$ by scalar multiplication.*

**Proof.** Let $S = W(m,n)$ with $(m,n) \neq (1,1)$ or $S = S(m,n)$ with $(m,n) \neq (1,2)$, with the principal grading. By Proposition 1.2($d$), $AutgrS \subset GL_m \times GL_n$. But the group on the right is contained in $AutgrS$ since it acts by automorphisms of $S$ via linear changes of indeterminates. It follows that $AutgrS \cong GL_m \times GL_n$.

Let $S = K(2k+1,n) = \prod_{j \geq -2} \mathfrak{g}_j$ with its principal grading. By Proposition 1.2($e$), $AutgrS \subset \bar{\mathbb{F}}^{\times} \cdot (Sp_{2k} \times O_n)$. But the group on the right is contained in $AutgrS$ since it acts by automorphisms of $S$ via linear changes of indeterminates, preserving the supercontact differential form $dx_{2k+1} + \sum_{i=1}^{k}(x_i dx_{k+i} - x_{k+i} dx_i) + \sum_{j=1}^{n} \xi_j d\xi_{n-j+1}$ up to multiplication by a non-zero number. It follows that $AutgrS \cong \bar{\mathbb{F}}^{\times} \cdot (Sp_{2k} \times O_n)$. Likewise, if $S = H(2k,n) = \prod_{j \geq -1} \mathfrak{g}_j$ with the principal grading, the group $\bar{\mathbb{F}}^{\times} \cdot (Sp_{2k} \times O_n)$ acts by automorphisms of $S$ via linear changes of indeterminates, preserving the supersymplectic differential form $\sum_{i=1}^{k} dp_i \wedge dq_i + \sum_{j=1}^{n} d\xi_j d\xi_{n-j+1}$ up to multiplication by a non-zero number. Hence again, $AutgrS \cong \bar{\mathbb{F}}^{\times} \cdot (Sp_{2k} \times O_n)$.

Consider the Lie superalgebras $S = HO(n,n)$, $SHO(n,n)$ with $n > 3$, $KO(n,n+1)$, or $SKO(n,n+1;\beta)$ with $n > 2$ or with $n = 2$ and $\beta \neq 0,1$, with their principal gradings. By Proposition 1.2($f$), $AutgrS \subset \bar{\mathbb{F}}^{\times} \cdot GL_n$. On the other hand, the group on the right is contained in $AutgrS$ since it acts by automorphisms of $S$ via linear changes of indeterminates, preserving the odd supersymplectic form $\sum_{i=1}^{n} dx_i d\xi_i$ and the volume form $v$ attached to the usual divergence up to multiplication by a non-zero number, if $S = HO(n,n)$ or $S = SHO(n,n)$ with $n > 3$, and preserving the odd supercontact form $d\xi_{n+1} + \sum_{i=1}^{n}(\xi_i dx_i + x_i d\xi_i)$ up to multiplication by an invertible function, and the volume form $v_\beta$ attached to the $\beta$-divergence up to multiplication by a non-zero number, if $S = KO(n,n+1)$ or $S = SKO(n,n+1;\beta)$ with $\beta \neq 0,1$ if $n = 2$. Therefore $AutgrS \cong \bar{\mathbb{F}}^{\times} \cdot GL_n$.

Let $S = SHO^{\sim}(n,n)$, with $n > 2$ even, and let $S_0$ be the principal subalgebra of $S$. The group $Aut(GrS)$ consists of the automorphisms of $SHO'(n,n)$ preserving its principal grading. By the same argument as for $SHO(n,n)$, $Aut(GrS) \cong \bar{\mathbb{F}}^{\times} \cdot GL_n$. The subgroup $AutgrS$ consists of the elements in $Aut(GrS)$ which can be lifted to automorphisms of $S$. Every element in $SL_n$ can be lifted to an automorphism of $S$, since it preserves the form $Fv$ defining the Lie superalgebra $S$. Besides, such automorphisms are inner and act on $S$ via linear changes of variables. On the contrary, the outer automorphisms of $SHO'(n,n)$ do not preserve the form $Fv$ for any $t \in \bar{\mathbb{F}}$, hence they cannot be lifted to any automorphism of $S$. It follows that $AutgrS \cong SL_n$. The argument for $S = SKO^{\sim}(n,n+1)$ is similar.

Consider $S = S(1,2) = \prod_{j \geq -2} \mathfrak{g}_j$ with the grading of type $(2|1,1)$. By Proposition 1.2($a$), $AutgrS \subset \bar{\mathbb{F}}^{\times} \cdot SO_4$. Notice that $\exp(ad(\mathfrak{g}_0)) \cong \bar{\mathbb{F}}^{\times} \cdot SL_2$, acting by automorphisms of $S$ via linear changes of indeterminates which preserve

the standard volume form $v$ up to multiplication by a non-zero number. Since the algebra of outer derivations of $S$ is isomorphic to $sl_2$, $AutgrS \cong \bar{\mathbb{F}}^\times \cdot SO_4$.

Consider the Lie superalgebra $S = SHO(3,3) = \prod_{j\geq -j} \mathfrak{g}_j$ with the grading of type $(2,2,2|1,1,1)$. By Proposition 1.2$(a)$, $AutgrS \subset \bar{\mathbb{F}}^\times \cdot (SL_3 \times SL_2)$. Notice that $\exp(ad(\mathfrak{g}_0)_{\bar{0}}) \cong SL_3$, acting by automorphisms of $S$ via linear changes of indeterminates, which preserve the odd supersymplectic form and the volume form $v$ up to multiplication by a non-zero number. Since the algebra of outer derivations of $S$ is isomorphic to $gl_2$, $AutgrS \cong \bar{\mathbb{F}}^\times \cdot (SL_3 \times SL_2)$.

Consider the Lie superalgebra $S = SKO(2,3;1) = \prod_{j\geq -1} \mathfrak{g}_j$, with its subprincipal grading. In this case $\mathfrak{g}_{-1} = \bar{\mathbb{F}}^{2|2}$ hence, by Proposition 2.1$(a)$, $AutgrS \subset GL_2 \times GL_2$. As we recalled in Section 1, $(\mathfrak{g}_0)_{\bar{0}} \cong sl_2 + \bar{\mathbb{F}}$, where $sl_2$ acts trivially on $(\mathfrak{g}_{-1})_{\bar{1}}$ and by the standard action on $(\mathfrak{g}_{-1})_{\bar{0}}$, and $\bar{\mathbb{F}}$ acts as the scalar $-2$ on $\mathfrak{g}_{-2}$. Besides, the algebra of outer derivations of $S$ is isomorphic to $sl_2$, it acts trivially on $(\mathfrak{g}_{-1})_{\bar{0}}$ and by the standard action on $(\mathfrak{g}_{-1})_{\bar{1}}$. It follows that $AutgrS \cong \bar{\mathbb{F}}^\times \cdot (SL_2 \times SL_2)$.

Consider the Lie superalgebra $S = E(1,6)$ with its principal grading. By Proposition 1.2$(e)$ with $k = 0$ and $n = 6$, $AutgrS \subset \bar{\mathbb{F}}^\times \cdot O_6$. Notice that $\exp(ad(\mathfrak{g}_0)_{\bar{0}}) \cong \bar{\mathbb{F}}^\times \cdot SO_6$ and the group $O_6/SO_6$ is generated by the change of variables $\xi_i \leftrightarrow \eta_i$ which is not an automorphism of $E(1,6)$, since it exchanges the submodules $\mathfrak{g}_1^+$ and $\mathfrak{g}_1^-$ of the 1-st graded component of $K(1,6)$ in its principal grading (cf. [2, §6]). Therefore $AutgrS \cong \bar{\mathbb{F}}^\times \cdot SO_6$.

Consider $S = E(3,6) = \prod_{j\geq -2} \mathfrak{g}_j$ with its principal grading. By Proposition 1.2$(b)$, $AutgrS \subset \bar{\mathbb{F}}^\times \cdot (SL_2 \times SL_3)$. Since $\exp(ad(\mathfrak{g}_0)_{\bar{0}}) \cong \bar{\mathbb{F}}^\times \cdot (SL_2 \times SL_3)$, the statement follows. The same argument holds for $S = E(3,8)$ in its principal grading.

Consider $S = E(4,4) = \prod_{j\geq -1} \mathfrak{g}_j$ with its principal grading. By Proposition 1.2$(g)$, $AutgrS \subset \bar{\mathbb{F}}^\times \cdot SL_4$. Since $\exp(ad(\mathfrak{g}_0)_{\bar{0}}) \cong GL_4$, equality holds.

Finally, consider $S = E(5,10) = \prod_{j\geq -2} \mathfrak{g}_j$ with its principal grading. By Proposition 1.2$(c)$, $AutgrS \subset \bar{\mathbb{F}}^\times \cdot SL_5$. Besides, $\exp(ad(\mathfrak{g}_0)) \cong SL_5$. Note that the Lie superalgebra $S$ has an outer derivation acting on $S = \prod_{j\geq -2} \mathfrak{g}_j$ as the grading operator. It follows that $AutgrS \cong GL_5$. □

**Corollary 4.3** *Let $S$ be a simple infinite-dimensional linearly compact Lie superalgebra over $\bar{\mathbb{F}}$. Then $AutS$ is the semidirect product of the group of inner automorphisms of $S$ and the finite-dimensional algebraic group $A$, described below:*

- $(a)$ *if $S = SHO^\sim(n,n)$ or $SKO^\sim(n,n+1)$, then $A = \{1\}$;*
- $(b)$ *if $S$ is a Lie algebra or $S = E(1,6)$, $E(3,6)$, $E(3,8)$, $E(4,4)$, $E(5,10)$, then $A \cong \bar{\mathbb{F}}^\times$;*
- $(c)$ *if $S = H(m,n)$ or $K(m,n)$ with $n > 0$, then $A \cong \bar{\mathbb{F}}^\times \times \mathbb{Z}_2$;*
- $(d)$ *if $S = W(m,n)$ with $(m,n) \neq (1,1)$ and $n > 0$, $S(m,n)$ with $m > 1$ and $n > 0$ or with $m = 1$ and $n$ odd, $HO(n,n)$, $SHO(n,n)$ with $n > 3$ even,*

$KO(n,n+1)$, $SKO(n,n+1;\beta)$ *with* $\beta \neq (n-2)/n, 1$, *or with* $n$ *even and* $\beta = (n-2)/n$, *or with* $n$ *odd and* $\beta = 1$, *then* $A \cong \bar{\mathbb{F}}^{\times 2}$*;*

(e) *if* $S = S(1,n)$, *with* $n > 2$ *even,* $SKO(n,n+1;(n-2)/n)$ *with* $n > 2$ *odd,* $SKO(n,n+1;1)$ *with* $n > 2$ *even, or* $SHO(n,n)$ *with* $n > 3$ *odd, then* $A \cong U \rtimes \bar{\mathbb{F}}^{\times 2}$ *where* $U$ *is a one-dimensional unipotent group;*

(f) *if* $S = S(1,2)$, *then* $A \cong \bar{\mathbb{F}}^{\times} \times SO_3$*;*

(g) *if* $S = SHO(3,3)$ *or* $SKO(2,3;1)$, *then* $A \cong \bar{\mathbb{F}}^{\times} \cdot SL_2$.

We shall now investigate the nature of all continuous automorphisms of each simple infinite-dimensional non-exceptional linearly compact Lie superalgebra $S$ over $\bar{\mathbb{F}}$.

**Lemma 4.4** *Consider a subalgebra* $L$ *of* $W(m,n)$ *and let* $D$ *be an even element of* $L$ *lying in the first member of a filtration of* $W(m,n)$. *Then* $D$ *lies in the Lie algebra of the group of changes of variables which map* $L$ *to itself.*

**Proof.** Let $D = \sum_i P_i \frac{\partial}{\partial x_i} + \sum_j Q_j \frac{\partial}{\partial \xi_j}$. Then $\exp tD$, when applied to $x_i$ and $\xi_j$, gives convergent series $S_i(t)$ and $R_j(t)$, respectively (in the linearly compact topology), hence the change of variables $x_i \to S_i(t)$, $\xi_j \to R_j(t)$ is a one-parameter group of automorphisms of $W(m,n)$ which preserves $L$. $\square$

**Theorem 4.5** *Let* $S \subset W(m,n)$ *be the defining embedding of a non-exceptional simple infinite-dimensional linearly compact Lie superalgebra. If* $S \neq S(1,2)$, $SHO(3,3)$, $SKO(2,3;1)$, *and* $S$ *is defined by an action on a volume form* $v$, *an even or odd supersymplectic form* $\omega_s$, *or an even or odd supercontact form* $\omega_c$, *then all continuous automorphisms of* $S$ *over* $\bar{\mathbb{F}}$ *are obtained by invertible changes of variables, multiplying* $v$ *and* $\omega_s$ *by a constant and* $\omega_c$ *by a function. If* $S = S(1,2)$, $SHO(3,3)$, *or* $SKO(2,3;1)$, *then all these changes of variables form a subgroup* $H$ *of* $AutS$ *of codimension one.*

**Proof.** Let $S = W(m,n)$ with $(m,n) \neq (1,1)$. Then $DerS = S$ hence, by Lemma 4.4, $AutfS$ consists of invertible changes of variables. Besides, $AutgrS$ consists of linear changes of variables, thus, by Proposition 2.1$(c)$, the statement for $W(m,n)$ follows.

Let now $S = S(m,n)$ with $(m,n) \neq (1,2)$. Then $DerS = CS'(m,n) \subset W(m,n)$. By Lemma 4.4, $AutfS$ lies in the group of changes of variables whose Lie algebra kills the volume form $v$ attached to the standard divergence. Hence these changes of variables preserve the form $v$. It is clear that all linear changes of variables multiply the volume form $v$ by a number, hence all of them are automorphisms of $S(m,n)$. Hence $AutS$ is the group of changes of variables which preserve the volume form up to multiplication by a non-zero number.

If $S = S(1,2)$, then, by the same argument as above, the inner automorphisms of $S$ and its automorphism $t^c$, where $c$ is the grading operator of $S$ with respect to its grading of type $(2|1,1)$, are induced by changes of variables which

preserve the volume form up to multiplication by a non-zero number. We recall that the algebra $\mathfrak{a}$ of outer derivations of $S$ is isomorphic to $sl_2$, with standard generators $e$, $f$, $h$, where $e = ad(\xi_1\xi_2\frac{\partial}{\partial x})$ and $h = ad(\xi_1\frac{\partial}{\partial\xi_1} + \xi_2\frac{\partial}{\partial\xi_2})$ (cf. [2, Remark 2.12]). As for $S(m,n)$, $t^h$, where $t \in \bar{\mathbb{F}}$, is obtained by a linear change of variables preserving the volume form up to multiplication by a non-zero number. Besides, the element $\xi_1\xi_2\frac{\partial}{\partial x}$ is contained in the first member of the principal filtration of $S'(1,2)$, thus it is obtained by a change of variables preserving the volume form up to multiplication by a non-zero number, by Lemma 4.4. On the other hand, the automorphism $\exp(f)$ cannot be induced by any change of variables, since it does not preserve the principal filtration of $S$.

The argument for all other non-exceptional Lie superalgebras is similar. $\Box$

**Remark 4.6** Let $S = S(1,2)$, $SHO(3,3)$, or $SKO(2,3;1)$. In all these cases the algebra of outer derivations contains $sl_2 = \langle e, h, f\rangle$. Denote by $U_-$ the one-parameter group of automorphisms $\exp(ad(tf))$, where $f$ is explicitely described in [2, Remarks 2.12, 2.37, 4.15]. Then $U_-$ is the "complementary" to $H$ subgroup in $AutS$, namely, for every $\varphi \in AutS$, either $\varphi \in U_-H$ or $\varphi \in U_-sH$ where $s$ is the reflection $s = \exp(e)\exp(-f)\exp(e)$.

# 5 $\mathbb{F}$-Forms

Let $\mathbb{F}$ be a field of characteristic zero and let $\bar{\mathbb{F}}$ be its algebraic closure.

**Definition 5.1** *Let $L$ be a Lie superalgebra over $\bar{\mathbb{F}}$. A Lie superalgebra $L^{\mathbb{F}}$ over $\mathbb{F}$ is called an $\mathbb{F}$-form of $L$ if $L^{\mathbb{F}} \otimes_{\mathbb{F}} \bar{\mathbb{F}} \cong L$.*

Denote by $Gal$ the Galois group of $\mathbb{F} \subset \bar{\mathbb{F}}$. Then $Gal$ acts on $AutL$ as follows:

$$\alpha.\varphi := \varphi^{\alpha} = \alpha\varphi\alpha^{-1}, \quad \alpha \in Gal, \ \varphi \in AutL.$$

To any $\mathbb{F}$-form $L^{\mathbb{F}}$ of $L$, i.e., to any isomorphism $\phi : L^{\mathbb{F}} \otimes_{\mathbb{F}} \bar{\mathbb{F}} \to L$, we can associate the map $\gamma_\alpha : Gal \to AutL$, $\alpha \mapsto \phi^\alpha\phi^{-1}$. The map $\gamma_\alpha$ satisfies the cocycle condition, i.e., $\gamma_{\alpha\beta} = \gamma_\beta^\alpha\gamma_\alpha$. Two cocycles $\gamma$ and $\delta$ are equivalent if and only if there exists an element $\psi \in AutL$ such that $\gamma_\alpha = (\psi^{-1})^\alpha\delta_\alpha\psi$. It follows that equivalent cocycles correspond to isomorphic $\mathbb{F}$-forms.

**Proposition 5.2** *The map $\phi \longmapsto \{\alpha \mapsto \phi^\alpha\phi^{-1}\}$ induces a bijection between the set of isomorphism classes of $\mathbb{F}$-forms of $L$ and $H^1(Gal, AutL)$.*

**Proof.** For a proof see [13, §4].

We recall the following standard result (cf. [14, §VII.2]):

**Proposition 5.3** *If $K$ is a group and $A$, $B$, $C$ are groups with an action of $K$ by automorphisms, related by an exact sequence:*

$$1 \to A \to B \to C \to 1,$$

*then there is a cohomology long exact sequence:*

$$1 \to H^0(K,A) \to H^0(K,B) \to H^0(K,C) \to H^1(K,A) \to H^1(K,B) \to H^1(K,C),$$

*where the first three maps are group homomorphisms, and the last three are maps of pointed sets.*

**Proposition 5.4** *Let $S$ be a simple infinite-dimensional linearly compact Lie superalgebra over $\bar{\mathbb{F}}$. Then the map $j : AutS \to AutgrS$ induces an embedding*

$$j_* : H^1(Gal, AutS) \longrightarrow H^1(Gal, AutgrS).$$

**Proof.** The same arguments as in [13, Proposition 4.2] show that $H^1(Gal, AutfS) = 0$. Then the statement follows from exact sequence (4) in Section 2 and Proposition 5.3. □

We recall the following well known results on Galois cohomology. All details can be found in [14, § X] and [15, § III Annexe].

**Theorem 5.5** *(a) $H^1(Gal, \bar{\mathbb{F}}^\times) = 1$;*

*(b) $H^1(Gal, GL_n(\bar{\mathbb{F}})) = 1$;*

*(c) $H^1(Gal, SL_n(\bar{\mathbb{F}})) = 1$;*

*(d) $H^1(Gal, Sp_n(\bar{\mathbb{F}})) = 1$;*

*(e) if $q$ is a quadratic form over $\mathbb{F}$, then there exists a bijection between $H^1(Gal, O_n(q, \bar{\mathbb{F}}))$ and the set of classes of $\mathbb{F}$-quadratic forms which are $\bar{\mathbb{F}}$-isomorphic to $q$;*

*(f) if $q$ is a quadratic form over $\mathbb{F}$, then there exists a bijection between $H^1(Gal, SO_n(q, \bar{\mathbb{F}}))$ and the set of classes of $\mathbb{F}$-quadratic forms $q'$ which are $\bar{\mathbb{F}}$-isomorphic to $q$ and such that $\det(q')/\det(q) \in (\mathbb{F}^\times)^2$.*

**Lemma 5.6** Let $G$ be an almost direct product over $\mathbb{F}$ of $\bar{\mathbb{F}}^\times$ and an algebraic group $G_1$, and let $C = \bar{\mathbb{F}}^\times \cap G_1(\bar{\mathbb{F}})$ be a cyclic group of order $k$. Then we have the following exact sequence:

$$1 \to \mathbb{F}^\times/(\mathbb{F}^\times)^k \to H^1(Gal, G_1) \to H^1(Gal, G) \to 1. \tag{5}$$

In particular, if $H^1(Gal, G_1) = 1$, then $H^1(Gal, G) = 1$.

**Proof.** We have the following exact sequence:

$$1 \to G_1 \to G \xrightarrow{\pi} \bar{\mathbb{F}}^\times \to 1,$$

where $\pi : G \to G/G_1 \cong \bar{\mathbb{F}}^\times/C \to \bar{\mathbb{F}}^\times$ is the composition of the canonical map of $G$ to $G/G_1$ and the map $x \mapsto x^k$ from $\bar{\mathbb{F}}^\times/C$ to $\bar{\mathbb{F}}^\times$. By Proposition 5.3, we get the following exact sequence:

$$1 \to \mathbb{F}^\times/C \to \mathbb{F}^\times \to H^1(Gal, G_1) \to H^1(Gal, G) \to 1.$$

This implies exact sequence (5). $\square$

We fix the $\mathbb{F}$-form $S^{\mathbb{F}}$ of each simple infinite-dimensional linearly compact Lie superalgebra $S$ over $\bar{\mathbb{F}}$, defined by the same conditions as in [2], but over $\mathbb{F}$ (in the case of $SKO(n, n+1; \beta)$ we need to assume that $\beta \in \mathbb{F}$). This is called the *split* $\mathbb{F}$-form of $S$. In more invariant terms, this $\mathbb{F}$-form is characterized by the condition that it contains a split maximal torus $T$ (i.e. $T$ is *ad*-diagonalizable over $F$ and $T \otimes_{\mathbb{F}} \bar{\mathbb{F}}$ is a maximal torus of $S$).

**Theorem 5.7** *Let $S$ be a simple infinite-dimensional linearly compact Lie superalgebra over $\bar{\mathbb{F}}$ not isomorphic to $H(m,n)$, $K(m,n)$, $E(1,6)$, or $S(1,2)$. Then any $\mathbb{F}$-form of $S$ is isomorphic to the split $\mathbb{F}$-form.*

**Proof.** It follows from Propositions 5.2, 5.4 and the description of the group $AutgrS$ given in Theorem 4.2, using Theorem 5.5 and Lemma 5.6. $\square$

**Remark 5.8** Let $S = H(2k,n)$ or $S = K(2k+1,n)$. Then, according to Table 1 and Lemma 5.6, we have the exact sequence

$$1 \to \mathbb{F}^{\times}/(\mathbb{F}^{\times})^2 \to H^1(Gal, Sp_{2k} \times O_n) \to H^1(Gal, AutgrS) \to 1.$$

Here and further, $Sp_{2k} = Sp_{2k}(\bar{\mathbb{F}})$ and $O_n \subset GL_n(\bar{\mathbb{F}})$ is the orthogonal group over $\bar{\mathbb{F}}$ which leaves invariant the quadratic form $\sum_{i=1}^{n} x_i x_{n-i+1}$. Since $H^1(Gal, G_1 \times G_2) \cong H^1(Gal, G_1) \times H^1(Gal, G_2)$, by Theorem 5.5(d), $H^1(Gal, Sp_{2k} \times O_n) \cong H^1(Gal, O_n)$, hence we have the exact sequence

$$1 \to \mathbb{F}^{\times}/(\mathbb{F}^{\times})^2 \to H^1(Gal, O_n) \to H^1(Gal, AutgrS) \to 1.$$

Given a non-degenerate quadratic form $q$ over $\mathbb{F}$ in $n$ indeterminates, associated to a symmetric matrix $c = (c_{ij})$, introduce the following supersymplectic and supercontact differential forms $\sigma_q$ and $\Sigma_q$:

$$\sigma_q = \sum_{i=1}^{k} dp_i \wedge dq_i + \sum_{i,j=1}^{n} c_{ij} d\xi_i d\xi_j,$$

$$\Sigma_q = dt + \sum_{i=1}^{k} (p_i dq_i - q_i dp_i) + \sum_{i,j=1}^{n} c_{ij} \xi_i d\xi_j.$$

**Theorem 5.9** $(a)$ *Any $\mathbb{F}$-form of the Lie superalgebra $S = H(2k,n)$ is isomorphic to one of the Lie superalgebras $H_q(2k,n) := \{X \in W(2k,n)^{\mathbb{F}} \mid X\sigma_q = 0\}$.*
$(b)$ *Any $\mathbb{F}$-form of the Lie superalgebra $S = K(2k+1,n)$ is isomorphic to one of the Lie superalgebras $K_q(2k+1,n) := \{X \in W(2k+1,n)^{\mathbb{F}} \mid X\Sigma_q = f\Sigma_q\}$.*

*Two such $\mathbb{F}$-forms $S_q$ and $S_{q'}$ of $S$ are isomorphic if and only if $q$ and $q'$ are equivalent non-degenerate quadratic forms over $\mathbb{F}$, up to multiplication by a non-zero scalar in $\mathbb{F}$.*

**Proof.** It is easy to see that every non-degenerate quadratic form $q$ over $\mathbb{F}$, with matrix $c = (c_{ij})$, gives rise to the $\mathbb{F}$-forms $H_q(2k, n)$ and $K_q(2k + 1, n)$ of the Lie superalgebras $S = H(2k, n)$ and $S = K(2k + 1, n)$, respectively, attached to the corresponding cocycles. By construction, equivalent quadratic forms give rise to isomorphic $\mathbb{F}$-forms of $S$. Besides, if $\lambda \in \mathbb{F}^\times$ and $q'$ is the quadratic form associated to the matrix $\lambda c$, then $S_q \cong S_{q'}$, and the isomorphism is given by the following change of variables:

$$p_i \mapsto \lambda^{-1} p_i, \quad q_i \mapsto q_i, \quad \xi_i \mapsto \xi_i, \quad \text{if } S = H(2k, n)$$

$$t \mapsto \lambda^{-1} t, \quad p_i \mapsto \lambda^{-1} p_i, \quad q_i \mapsto q_i, \quad \xi_i \mapsto \xi_i, \quad \text{if } S = K(2k + 1, n).$$

The $\mathbb{F}$-forms $S_q$ exhaust all $\mathbb{F}$-forms of the Lie superalgebra $S$, due to Proposition 5.2, Theorem 4.2, Remark 5.8 and Theorem 5.5($e$). □

**Example 5.10** Consider the $\mathbb{F}$-form $K_q(1, 6)$ of $K(1, 6)$ corresponding to the supercontact form $\Sigma_q = dt + \sum_{i=1}^{6} c_{ij} \xi_i d\xi_j$. Then the principal grading of $K(1, 6)$ induces an irreducible grading on $K_q(1, 6)$: $K_q(1, 6) = \prod_{j \geq -2} \mathfrak{g}_j$, where $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathbb{F}$, $\mathfrak{h}$ is an $\mathbb{F}$-form of $so_6(\bar{\mathbb{F}})$, $\mathfrak{g}_{-1} \cong \mathbb{F}^6$, and $\mathfrak{g}_1 = \mathfrak{g}_{-1}^* \oplus \Lambda^3(\mathbb{F}^6)$.

Let $d$ be the discriminant of the quadratic form $q$. If $-d \in (\mathbb{F}^\times)^2$, then the $\mathfrak{g}_0$-module $\Lambda^3(\mathbb{F}^6)$ is not irreducible, and decomposes over $\mathbb{F}$ into the direct sum of two $\mathfrak{g}_0$-submodules $\mathfrak{g}_1^+$ and $\mathfrak{g}_1^-$, which are the eigenspaces of the Hodge operator $^*$, see Example 6.2 below (they are obtained from one another by an automorphism of $\mathfrak{g}_0$). It follows that we can define an $\mathbb{F}$-form $E_q(1, 6)$ of the Lie superalgebra $E(1, 6)$ by repeating the same construction as the one described in Section 1, namely, $E_q(1, 6)$ will be the graded subalgebra of $K_q(1, 6)$ generated by $\mathfrak{g}_{-1} + \mathfrak{g}_0 + (\mathfrak{g}_{-1}^* + \mathfrak{g}_1^+)$.

**Theorem 5.11** *Any $\mathbb{F}$-form of the Lie superalgebra $S = E(1, 6)$ is isomorphic to one of the Lie superalgebras $E_q(1, 6)$ constructed in Example 5.10, where q is a non-degenerate quadratic form over $\mathbb{F}$ in six indeterminates, with discriminant $d \in -(\mathbb{F}^\times)^2$.*

*Two such $\mathbb{F}$-forms $E_q(1, 6)$ and $E_{q'}(1, 6)$ of $E(1, 6)$ are isomorphic if and only if the quadratic forms $q$ and $q'$ are equivalent, up to multiplication by a non-zero scalar in $\mathbb{F}$.*

**Proof.** By Lemma 5.6 and Theorem 4.2($h$), we have the exact sequence

$$1 \to \mathbb{F}^\times / (\mathbb{F}^\times)^2 \to H^1(Gal, SO_6) \to H^1(Gal, AutgrS) \to 1.$$

The statement follows, due to Proposition 5.2, Theorem 5.5($f$) and the proof of Theorem 5.9. □

**Remark 5.12** Consider the Lie superalgebra $K(1, 4) = \prod_{j \geq -2} \mathfrak{g}_j$ over $\bar{\mathbb{F}}$ with respect to its principal grading. Then: $\mathfrak{g}_0 \cong cso_4 = so_4 + \bar{\mathbb{F}}t$, $\mathfrak{g}_{-1} \cong \bar{\mathbb{F}}^4$ and $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] \cong \bar{\mathbb{F}}$. Besides, $\mathfrak{g}_1 = V_1 \oplus V_{-1}$, where for every $\lambda = \pm 1$, $V_\lambda$ is isomorphic to the standard $so_4$-module, $[V_\lambda, V_\lambda]$ is isomorphic to the trivial $so_4$-module $\bar{\mathbb{F}}$, and $\mathfrak{g}_{-2} + \mathfrak{g}_{-1} + gl_2 + V_\lambda + [V_\lambda, V_\lambda] \cong sl(2, 2)/\bar{\mathbb{F}}$. Finally, $\mathfrak{g}_2 =$

$\bar{\mathbb{F}} \oplus so_4 \oplus \bar{\mathbb{F}}$, where $so_4$ and $\bar{\mathbb{F}}$ denote the adjoint and the trivial $so_4$-module, respectively. Here $t$ acts as the grading operator.

The Lie superalgebra $S(1,2)$ over $\bar{\mathbb{F}}$ is the subalgebra of $K(1,4)$ generated by $\mathfrak{h}_{-1} \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$ where $\mathfrak{h}_{-1} = \mathfrak{g}_{-1}$, $\mathfrak{h}_0 = sl_2 + \bar{\mathbb{F}}t$, $\mathfrak{h}_1 = V_1$ and $\mathfrak{h}_2 = \bar{\mathbb{F}} + sl_2$, where $sl_2$ denotes the adjoint $sl_2$-module (see also [2, Remark 2.33]).

**Example 5.13** Consider a Lie superalgebra $\mathfrak{g}_-$ over $\mathbb{F}$ with consistent $\mathbb{Z}$-grading $\mathfrak{g}_- = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0$, where $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathbb{F}$, $\mathfrak{h}$ is an $\mathbb{F}$-form of $sl_2$, $\mathfrak{g}_{-2} = \mathbb{F}z$, and $\mathfrak{g}_{-1}$ is a four-dimensional $[\mathfrak{g}_0, \mathfrak{g}_0]$-module such that $\mathfrak{g}_{-1} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$ is the direct sum of two copies of the standard $sl_2$-module, and where the bracket in $\mathfrak{g}_{-1}$ is defined as follows:

$$[a,b] = q(a,b)z,$$

where $q$ is a non-degenerate bilinear form on $\mathfrak{g}_{-1}$ over $\mathbb{F}$, which is symmetric, i.e. $q(a,b) = q(b,a)$. Such a superalgebra $\mathfrak{g}_-$ exists if and only if the discriminant $d$ of the quadratic form $q$ lies in $(\mathbb{F}^\times)^2$. Let $S_q(1,2)$ denote the full prolongation of $\mathfrak{g}_-$ over $\mathbb{F}$ (see [4, §1.6]). Then $S_q(1,2)$ is an $\mathbb{F}$-form of $S(1,2)$.

**Theorem 5.14** *Any $\mathbb{F}$-form of the Lie superalgebra $S = S(1,2)$ is isomorphic to one of the Lie superalgebras $S_q(1,2)$ constructed in Example 5.13, where $q$ is a non-degenerate quadratic form over $\mathbb{F}$ in four indeterminates, with discriminant $d \in (\mathbb{F}^\times)^2$.*

*Two such $\mathbb{F}$-forms $S_q(1,2)$ and $S_{q'}(1,2)$ of $S(1,2)$ are isomorphic if and only if the quadratic forms $q$ and $q'$ are equivalent, up to multiplication by a non-zero scalar in $\mathbb{F}$.*

**Proof.** By Lemma 5.6 and Theorem 4.2, we have the exact sequence

$$1 \to \mathbb{F}^\times/(\mathbb{F}^\times)^2 \to H^1(Gal, SO_4) \to H^1(Gal, AutgrS) \to 1.$$

The statement follows, due to Proposition 5.2, Theorem 5.5($f$) and the proof of Theorem 5.9. □

We summarize the results of this section in the following theorem:

**Theorem 5.15** *Let $S$ be a simple infinite-dimensional linearly compact Lie superalgebra over $\bar{\mathbb{F}}$. If $S$ is not isomorphic to $H(m,n)$, $K(m,n)$, $E(1,6)$, or $S(1,2)$, then the split $\mathbb{F}$-form $S^{\mathbb{F}}$ is, up to isomorphism, the unique $\mathbb{F}$-form of $S$. In the remaining four cases, all $\mathbb{F}$-forms of $S$ are, up to isomorphism, as follows:*

($a$) *the Lie superalgebras $H_q(m,n) := \{X \in W(m,n)^{\mathbb{F}} \mid X\sigma_q = 0\}$ where $\sigma_q$ is a supersymplectic differential form over $\mathbb{F}$, if $S = H(m,n)$;*

($b$) *the Lie superalgebras $K_q(m,n) := \{X \in W(m,n)^{\mathbb{F}} \mid X\Sigma_q = f\Sigma_q\}$ where $\Sigma_q$ is a supercontact differential form over $\mathbb{F}$, if $S = K(m,n)$;*

($c$) *the Lie superalgebras $E_q(1,6)$ constructed in Example 5.10, where $q$ is a non-degenerate quadratic form over $\mathbb{F}$ in six indeterminates with discriminant $d \in -(\mathbb{F}^\times)^2$, if $S = E(1,6)$;*

(*d*) *the Lie superalgebras* $S_q(1,2)$ *constructed in Example 5.13, where* $q$ *is a non-degenerate quadratic form over* $\mathbb{F}$ *in four indeterminates with discriminant* $d \in (\mathbb{F}^\times)^2$*, if* $S = S(1,2)$*.*

*The isomorphisms between these* $\mathbb{F}$*-forms are described in Theorems 5.9, 5.11, 5.14.*

**Remark 5.16** It follows immediately from Theorem 5.15 that a simple infinite-dimensional linearly compact Lie superalgebra $S$ over $\mathbb{C}$ has, up to isomorphism, one real form if $S$ is not isomorphic to $H(m,n)$, $K(m,n)$, $E(1,6)$, or $S(1,2)$, two real forms if $S$ is isomorphic to $E(1,6)$ or $S(1,2)$, and $[n/2]+1$ real forms if $S$ is isomorphic to $H(m,n)$ or $K(m,n)$.

# 6 Finite Simple Lie Conformal Superalgebras

In this section we use the theory of Lie conformal superalgebras in order to give an explicit construction of all non-split forms of all simple infinite-dimensional linearly compact Lie superalgebras. In conclusion of the section, we give the related classification of all $\mathbb{F}$-forms of all simple finite Lie conformal superalgebras.

We briefly recall the definition of a Lie conformal superalgebra and of its annihilation algebra. For notation, definitions and results on Lie conformal superalgebras we refer to [5], [6] and [11].

A Lie conformal superalgebra $R$ over $\mathbb{F}$ is a left $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{F}[\partial]$-module endowed with an $\mathbb{F}$-linear map, called the $\lambda$-bracket,

$$R \otimes R \to \mathbb{F}[\lambda] \otimes R, \ a \otimes b \mapsto [a_\lambda b],$$

satisfying the axioms of sesquilinearity, skew-commutativity, and the Jacobi identity. One writes $[a_\lambda b] = \sum_{n \in \mathbb{Z}_+} \frac{\lambda^n}{n!}(a_{(n)}b)$; the coefficient $(a_{(n)}b)$ is called the $n$-th product of $a$ and $b$. A Lie conformal superalgebra $R$ is called *finite* if it is finitely generated as an $\mathbb{F}[\partial]$-module.

Given a finite Lie conformal superalgebra $R$, we can associate to it a linearly compact Lie superalgebra $L(R)$ as follows. Consider the Lie conformal superalgebra $R[[t]]$, where $t$ is an even indeterminate, the $\partial$-action is defined by $\partial + \partial_t$, and the $n$-th products are defined by:

$$a(t)_{(n)}b(t) := \sum_{j \geq 0} (\partial_t^j a(t))_{(n+j)} b(t)/j!,$$

where $a(t)$,$b(t) \in R[[t]]$, and the $n$-th products on the right are extended from $R$ to $R[[t]]$ by bilinearity. Then $(\partial + \partial_t)R[[t]]$ is a two-sided ideal of $R[[t]]$ with respect to 0-th product, and this product induces a Lie superalgebra bracket on $L(R) := R[[t]]/(\partial + \partial_t)R[[t]]$. The linearly compact Lie superalgebra $L(R)$ is called the *annihilation algebra* of $R$.

For the classification of finite simple Lie conformal superalgebras over an algebraically closed field $\bar{F}$ of characteristic zero we refer to [6]. The list consists

of four series ($N \in \mathbb{Z}_+$): $W_N$, $S_{N+2,a}$, $\tilde{S}_{N+2}$, $K_N$ ($N \neq 4$), $K'_4$, the exceptional Lie conformal superalgebra $CK_6$ of rank 32, and $Cur\mathfrak{s}$, where $\mathfrak{s}$ is a simple finite-dimensional Lie superalgebra.

**Example 6.1** Let $V$ be an $N$-dimensional vector space over $\mathbb{F}$ with a non-degenerate symmetric bilinear form $q$. The Lie conformal superalgebra $K_{N,q}$ associated to $V$ is $\mathbb{F}[\partial]\Lambda(V)$ with $\lambda$-bracket:

$$(6) \qquad [A_\lambda B] = (\frac{r}{2} - 1)\partial(AB) + (-1)^r \frac{1}{2}\sum_{j=1}^{N}(i_{a_j}A)(i_{b_j}B) + \lambda(\frac{r+s}{2} - 2)AB,$$

where $A, B \in \Lambda(V)$, $r = \deg(A)$, $s = \deg(B)$, $a_i, b_i \in V$, $q(a_i, b_j) = \delta_{i,j}$, and $i_a$, for $a \in V$, denotes the contraction with $a$, i.e., $i_a$ is the odd derivation of $\Lambda(V)$ defined by: $i_a(b) = q(a, b)$ for $b \in V$ (cf [6, Example 3.8]). The annihilation algebra of $K_{N,q}$ is isomorphic to the Lie superalgebra $K_q(1, N)$ defined in Theorem 5.9($b$). The Lie conformal superalgebra $K_{N,q}$ is an $\mathbb{F}$-form of the finite simple Lie conformal superalgebra $K_N$.

**Example 6.2** Let $V$ be an $N$-dimensional vector space over $\mathbb{F}$ with a non-degenerate symmetric bilinear form $q$, and let $K_{N,q}$ be the Lie conformal superalgebra over $\mathbb{F}$ constructed in Example 6.1. Choose a basis $\xi_1, \dots, \xi_N$ of $V$, and let $^*$ denote the Hodge star operator on $V$ associated to the form $q$, i.e.,

$$(\xi_{j_1} \wedge \xi_{j_2} \wedge \cdots \wedge \xi_{j_k})^* = i_{\xi_{j_1}} i_{\xi_{j_2}} \dots i_{\xi_{j_k}} (\xi_1 \wedge \cdots \wedge \xi_N).$$

It is easy to check that, for every $a \in \Lambda(V)$, $(a^*)^* = (-1)^{N(N-1)/2} \det(q)a$.

Let $N = 6$ and choose $\alpha \in \mathbb{F}$ such that $\alpha^2 = -1/det(q)$. Consider the following elements in $\Lambda(V)$:

$$-1 + \alpha\partial^3 1^*,\ \xi_i\xi_j + \alpha\partial(\xi_i\xi_j)^*,\ \xi_i - \alpha\partial^2\xi_i^*,\ \xi_i\xi_j\xi_k + \alpha(\xi_i\xi_j\xi_k)^*.$$

It is easy to check that the $\mathbb{F}[\partial]$-span of these elements is closed under $\lambda$-bracket (6), hence they form an $\mathbb{F}$-form $CK_{6,q}$ of the Lie conformal subalgebra $CK_6$ of $K_6$ (cf [3, Theorem 3.1]).

Likewise, if $N = 4$ and $\beta^2 = 1/det(q)$, the $\mathbb{F}[\partial]$-span of the elements:

$$-1 - \beta\partial^2 1^*,\ \xi_i\xi_j - \beta(\xi_i\xi_j)^*,\ \xi_i + \beta\partial\xi_i^*$$

is closed under $\lambda$-bracket (6). It follows that these elements form a subalgebra $S_{2,q}$ of $K_{4,q}$, which is an $\mathbb{F}$-form of the Lie conformal superalgebra $S_{2,0}$ (cf. [3, Remark p. 225]).

**Remark 6.3** The annihilation algebras of the Lie conformal superalgebras $CK_{6,q}$ and $S_{2,q}$, constructed in Examples 6.1 and 6.2, are the Lie superalgebras $E_q(1, 6)$ and $S_q(1, 2)$, constructed in Examples 5.10 and 5.13, respectively. Due to Theorems 5.11 and 5.14, Example 6.2 provides an explicit construction of all $\mathbb{F}$-forms of the Lie superalgebras $E(1, 6)$ and $S(1, 2)$.

We conclude by classifying all $\mathbb{F}$-forms of all simple finite Lie conformal superalgebras over $\bar{F}$. The following theorem can be derived from [7, Remark 3.1].

**Theorem 6.4** *Let $R$ be a simple finite Lie conformal superalgebra over $\bar{\mathbb{F}}$. If $R$ is not isomorphic to $S_{2,0}$, $K_N$, $K_4'$, $CK_6$ or $Cur\mathfrak{s}$, then there exists, up to isomorphism, a unique $\mathbb{F}$-form of $R$ (in the case $R = S_{N,a}$, we have to assume that $a \in \mathbb{F}$ for such a form to exist). In the remaining cases, all $\mathbb{F}$-forms of $R$ are as follows:*

- *the Lie conformal superalgebras $K_{N,q}$ if $R$ is isomorphic to $K_N$;*
- *the Lie conformal superalgebras $CK_{6,q}$ if $R$ is isomorphic to $CK_6$;*
- *the Lie conformal superalgebras $S_{2,q}$ if $R$ is isomorphic to $S_{2,0}$;*
- *the derived algebras of the Lie conformal superalgebras $K_{4,q}$ if $R$ is isomorphic to $K_4'$;*
- *the Lie conformal superalgebras $Cur\mathfrak{s}^{\mathbb{F}}$, where $\mathfrak{s}^{\mathbb{F}}$ is an $\mathbb{F}$-form of the Lie superalgebra $\mathfrak{s}$, if $R$ is isomorphic to $Cur\mathfrak{s}$.*


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# References


[1] N. Cantarini, S.-J. Cheng, V. G. Kac *Errata: Structure of some $\mathbb{Z}$-graded Lie superalgebras of vector fields,* Transf. Groups **9** (2004), 399-400.

[2] N. Cantarini, V. G. Kac *Infinite dimensional primitive linearly compact Lie superalgebras,* math. QA/0511424.

[3] S.-J. Cheng, V. G. Kac *A new $N = 6$ superconformal algebra,* Comm. Math. Phys. **186** (1997), 219–231.

[4] S.-J. Cheng, V. G. Kac *Structure of some $\mathbb{Z}$-graded Lie superalgebras of vector fields,* Transf. Groups **4** (1999), 219–272.

[5] A. D'Andrea, V. G. Kac *Structure theory of finite conformal algebras,* Selecta Math. (N.S.) **4** (1998), 377–418.

[6] D. Fattori, V. G. Kac *Classification of finite simple Lie conformal superalgebras,* J. Algebra **258** (2002), 23–59.

[7] D. Fattori, V. G. Kac, A. Retakh *Structure theory of finite Lie conformal superalgebras,* In "Lie theory and its applications to physics", eds H.-D. Doebner and V. V. Dobrev, World Sci., (2004) 27–63.

[8] V. W. Guillemin *Infinite-dimensional primitive Lie algebras,* J. Diff. Geom. **4** (1970), 257–282.

[9] V. G. Kac *Lie superalgebras,* Adv. Math. **26** (1977), 8–96.

[10] V. G. Kac *Classification of infinite-dimensional simple linearly compact Lie superalgebras,* Adv . Math. **139** (1998), 1–55.

[11] V. G. Kac *Vertex Algebras for Beginners,* Second Edition, University Lecture Series **10**, AMS, 1998.

[12] V. G. Kac *Classification of infinite-dimensional simple groups of supersymmetries and quantum field theory,* GAFA, Geom. Funct. Anal. Special Volume GAFA2000 (2000), 162–183.

[13] A. N. Rudakov *Groups of automorphisms of infinite-dimensional simple Lie algebras,* Math. USSR-Izvestija Vol. **3(4)** (1969), 707–722.

[14] J.P. Serre *Corps locaux,* Hermann, Paris, 1962.

[15] J.P. Serre *Cohomologie Galoisienne. Cinquième édition,* Lecture Notes in Mathematics **5**, Springer-Verlag, Berlin-Heidelberg, 1994.

[16] I. Shchepochkina *The five exceptional simple Lie superalgebras of vector fields and their fourteen regradings,* Repr. Theory 3 (1999), 373–415.